

COUNTABLE DENSE HOMOGENEITY OF DEFINABLE SPACES

MICHAEL HRUŠÁK AND BEATRIZ ZAMORA AVILÉS

June 13, 2003

ABSTRACT. We investigate which definable separable metric spaces are countable dense homogeneous (CDH). We prove that a Borel CDH space is completely metrizable and give a complete list of zero-dimensional Borel CDH spaces. We also show that for a Borel $X \subseteq 2^\omega$ the following are equivalent: (1) X is G_δ in 2^ω , (2) X^ω is CDH and (3) X^ω is homeomorphic to 2^ω or to ω^ω . Assuming the Axiom of Projective Determinacy the results extend to all projective sets and under the Axiom of Determinacy to all separable metric spaces. In particular, modulo large cardinal assumption it is relatively consistent with ZF that all CDH separable metric spaces are completely metrizable. We also answer a question of Steprāns and Zhou by showing that $\mathfrak{p} = \min\{\kappa : 2^\kappa \text{ is not CDH}\}$.

0. INTRODUCTION

A separable topological space X is *countable dense homogeneous (CDH)* if given any two countable dense subsets $D, D' \subseteq X$ there is a homeomorphism h of X such that $h[D] = D'$. The first result in this area is due to Cantor, who, in effect, showed that the reals are CDH. Fréchet [Fr] and Brower [Br], independently, proved that the same is true for the n -dimensional Euclidean space \mathbb{R}^n . In 1962, Fort [Fo] proved that the Hilbert cube is also CDH.

Systematic study of CDH spaces was initiated by Bennett [Be] in 1972. Since then a number of papers were published on the topic, most of which are mentioned in the references. The focus remained on separable metric spaces. Under some set-theoretic assumptions like the Continuum Hypothesis or Martin's Axiom a variety of examples of countable dense homogeneous metric spaces were constructed: Assuming CH Fitzpatrick and Zhou constructed a CDH Bernstein subset of \mathbb{R}^n and a CDH subset of \mathbb{R} which is meager in itself; Baldwin and Beaudoin constructed Bernstein subset of \mathbb{R} under Martin's Axiom for countable partial orders.

In this paper we are concerned mostly with countable dense homogeneity of definable separable metric spaces. Our principal result states that every analytic CDH space is completely Baire. We use it to give a complete list of zero-dimensional Borel CDH spaces and to show that for a Borel $X \subseteq 2^\omega$ the following are equivalent:

Key words and phrases: Countable dense homogeneous, Borel, Baire

2000 Mathematics Subject Classification: 54E52, 54H05, 03E15

The first author's research was supported partially by grant GAČR 201/03/0933 and by a PAPIIT grant IN108802-2 and CONACYT grant 40057-F.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

(1) X is G_δ in 2^ω , (2) X^ω is CDH and (3) X^ω is homeomorphic to 2^ω or to ω^ω . These provide partial answers to the following problems of [FZ3]:

387. For which 0-dimensional subsets of \mathbb{R} is X^ω homogeneous? CDH?

and

389. Does there exist a CDH metric space that is not completely metrizable?

1. DESCRIPTIVE SET THEORY

In this section we review some of the classical results of descriptive set theory. For proofs and further reference consult e.g. [Ke]. Recall that a separable completely metrizable space is called a *Polish* space. We call a separable metric space *Borel*, if it is Borel in its completion. A separable metric space is *analytic* if it is a continuous image of the Baire space ω^ω . A space is *co-analytic* if it is a complement of an analytic subspace of some Polish space. Recall that a space is Borel if and only if it is both analytic and co-analytic. This is an old result of Souslin as is the following:

Theorem 1.1. *Every analytic space contains a homeomorphic copy of 2^ω .*

Recall that a subset A of a Polish space X is said to have the *Baire property* if there is an open set $U \subseteq X$ such that the symmetric difference $A \Delta U$ is meager in X .

Theorem 1.2. *Every analytic subspace of a Polish space has the Baire property.*

A topological space X is *Baire* if the complement of every meager subset of X is dense in X . Note that being Baire and having the Baire property are quite different notions. We will use the following corollary of Theorem 1.2:

Theorem 1.3. *Every analytic Baire space has a dense completely metrizable subspace.*

Proof. Let X be an analytic Baire space and let \bar{X} be its completion. By Theorem 1.2 there is an open set $U \subseteq \bar{X}$ such that $X \Delta U$ is meager in \bar{X} . That is $X \Delta U = \bigcup_{n \in \omega} F_n$, where each F_n is nowhere dense in \bar{X} . Note that U is a dense open subset of \bar{X} . Let $G = U \setminus \bigcup_{n \in \omega} \bar{F}_n$. Then G is completely metrizable as it is G_δ in \bar{X} , and G is a dense subset of X as X is Baire. \square

A topological space X is *completely Baire* if all of its closed subspaces are Baire. The following theorem is due to Hurewicz (see [Ke]).

Theorem 1.4. *Every co-analytic completely Baire space is completely metrizable.*

Under the Axiom of Projective Determinacy (PD) all of the above theorems hold for all projective sets. Similarly under the Axiom of Determinacy (AD) they hold for all separable metric spaces. For proof of the analogues of Theorems 1.1 and 1.2 (and hence also 1.3) in this context see e.g Theorem 27.9 of [Ka]. The fact that the variants of the Theorem 1.4 hold follows from the proof of Theorem 4 of [KLW].

The following characterization of zero-dimensional Polish spaces can be found in [Ke] and [vM2]

Theorem 1.5. (i) Every zero-dimensional separable compact completely metrizable space without isolated points is homeomorphic to 2^ω .
(ii) Every zero-dimensional separable locally compact non-compact completely metrizable space without isolated points is homeomorphic to $2^\omega \setminus \{0\}$.
(i) Every zero-dimensional separable completely metrizable space without isolated points in which all compact sets are nowhere dense is homeomorphic to 2^ω .

2. ANALYTIC CDH SPACES

In the article *Some Open Problems in Densely homogeneous spaces* of the *Open problems in topology* Fitzpatrick and Zhou ask (Question 389.) whether there is a CDH metric space which is not completely metrizable. We answer this question in the negative for Borel spaces. The following simple lemma ([FZ2]) will be used many times in what follows.

Lemma 2.1. A separable metric space X without isolated points is meager in itself if and only if there is a countable dense $D \subseteq X$ which is G_δ in X .

Proof. The reverse implication is obvious. For the forward implication let $X = \bigcup_{n \in \omega} F_n$, where each F_n is a closed nowhere dense subset of X . Enumerate a basis for the topology of X as $\{U_n : n \in \omega\}$ and recursively pick $x_n \in U_n \setminus \bigcup_{m \leq n} F_m$. Set $D = \{x_n : n \in \omega\}$. D is obviously a countable dense subset of X . To see that it is G_δ in X note that D intersects each F_n in a finite set, hence $X \setminus D = \bigcup_{n \in \omega} (F_n \setminus D)$ is F_σ in X . \square

Next we prove a decomposition lemma for CDH spaces.

Lemma 2.2. Every CDH space X can be written as a disjoint topological sum $X = I \oplus L \oplus R$, where I is the set of isolated points in X , L is locally compact without isolated points and R has the property that every compact subset of R is nowhere dense in R .

Proof. First we show that the set I of all isolated points of X is clopen in X . Note that I is countable as X is separable. If I is not closed, pick $x \in \bar{I} \setminus I$ and a set $C \subseteq X \setminus \bar{I}$ countable dense in $X \setminus \bar{I}$. Let $D_0 = I \cup C$ and $D_1 = D_0 \cup \{x\}$. The sets D_0 and D_1 are then countable dense subsets of X and we reach a contradiction by noting that there is no homeomorphism of X sending D_1 to D_0 , for x is not isolated but every neighborhood of x contains an isolated point, whereas all points in D_0 are either isolated or have a neighborhood which does not contain any isolated points.

Let $Y = X \setminus I$. Consider

$L = \{x \in Y : \exists U \subseteq Y \text{ locally compact neighborhood of } x\}$ and

$R = \{x \in Y : \exists U \subseteq Y \text{ neighborhood of } x, \text{ s. t. } \forall K \subseteq U \text{ compact } \text{int}(K) = \emptyset\}$.

Obviously L and R are disjoint open subsets of Y . To finish the proof it suffices to show that $Y = L \cup R$. First note that $L \cup R$ is dense in Y , as if $x \in Y \setminus (L \cup R)$ then i.p. $x \in Y \setminus R$, which implies that for every $U \subseteq X$ neighborhood of x , there is a $K \subseteq U$ compact such that $\text{int}(K) \neq \emptyset$, hence $x \in \bar{L}$.

Now, suppose that $Y \setminus (L \cup R) \neq \emptyset$. Pick $x \in Y \setminus (L \cup R)$ and a countable dense $D_0 \subseteq L \cup R$ and let $D_1 = D_0 \cup \{x\}$. Again, D_0 and D_1 are clearly countable in

X and there is no homeomorphism h of X sending D_1 to D_0 as then $h(x) \in L$ or $h(x) \in R$ but $x \notin L \cup R$. \square

Theorem 2.3. *Every analytic CDH space X is completely Baire.*

Proof. By Lemma 2.2 we can assume that X has no isolated points.

Claim 1. Every open subset of X is uncountable.

Assume not, that is $V = \bigcup\{U : U \text{ is a countable open subset of } X\}$ is not empty. Then V is itself a countable open set. Choose C a countable dense subset of $X \setminus V$ and $x \in V$. Let $D_0 = C \cup V$ and $D_1 = C \cup V \setminus \{x\}$. The sets D_0 and D_1 are then countable dense subsets of X . As X is CDH there is a homeomorphism h of X such that $h[D_1] = D_0$. Then, however, $h(x) \notin V$ and, unlike x , $h(x)$ does not have a countable neighborhood which contradicts the fact that h is a homeomorphism.

Claim 2. X is Baire.

Suppose it is not the case. That means that there is an open set $U \subseteq X$ which is meager in itself. By Lemma 2.1 there is a $C \subseteq U$ countable dense in U which is G_δ in U . Let D_0 be a countable dense subset of X such that $D_0 \cap U = C$.

Let $\{U_n : n \in \omega\}$ be an enumeration of some countable basis for the topology on X . By Claim 1, each U_n is uncountable, as every open subset of an analytic space is itself analytic, by Theorem 1.1, each U_n contains a subset F_n homeomorphic to 2^ω . Choose, for every $n \in \omega$, a countable $C_n \subseteq F_n$ dense in F_n and set $D_1 = \bigcup_{n \in \omega} C_n$. The set D_1 is then a countable dense subset of X .

Note that $D_1 \cap V$ is not G_δ in V for any open set $V \subseteq X$. To see this let V be an open subset of X . There is an $n \in \omega$ such that $U_n \subseteq V$, hence $F_n \subseteq V$. If $D_1 \cap V$ were G_δ in V , then $D_1 \cap F_n$ would be G_δ in F_n . As $C_n \subseteq D_1 \cap F_n$ it follows that $D_1 \cap F_n$ is dense in F_n . Lemma 2.1 then implies that F_n is meager in itself which contradicts the Baire Category Theorem for 2^ω .

To finish the proof of the claim it suffices to notice that the countable dense sets D_0 and D_1 have different (relative) topological properties in X hence there is no homeomorphism of X sending one to the other, which contradicts the fact that X is CDH.

Now we are ready to show that X is completely Baire. By Claim 2 and Theorem 1.3, there is a completely metrizable $G \subseteq X$ which is dense in X . Let D_0 be any countable dense subset of G (and consequently also a dense subset of X .) Note that D_0 has the property that if $E \subseteq D_0$ has no isolated points then E is not G_δ in \bar{E} , for if E were G_δ in \bar{E} then E would be G_δ in $\bar{E} \cap G$, but $\bar{E} \cap G$ is a G_δ subset of G , hence, is completely metrizable. However, by Baire Category Theorem this does not happen.

Aiming toward a contradiction again, assume that X is not completely Baire. That is, there is a closed set $F \subseteq X$ which is meager in itself. By Lemma 2.1 there is a countable dense $C \subseteq F$ which is G_δ in F . Let $D_1 = C \cup (D_0 \setminus F)$. The set D_1 is clearly a countable dense subset of X and has the property that there is a subset of it without isolated points which is G_δ in its closure (C being a witness to this.)

So, again, the countable dense sets D_0 and D_1 have different (relative) topological properties in X hence there is no homeomorphism of X sending one to the other contradicting the countable dense homogeneity of X . \square

Corollary 2.4. *Every Borel CDH space X is completely metrizable.*

Proof. Follows directly from Theorem 2.3 and Theorem 1.4. \square

Corollary 2.5. *Let X be a zero-dimensional Borel CDH space without isolated points. Then X is homeomorphic to one of the following five spaces: 2^ω , ω^ω , $2^\omega \setminus \{0\}$, $\omega^\omega \oplus 2^\omega$ and $\omega^\omega \oplus 2^\omega \setminus \{0\}$.*

Proof. By the previous corollary X is completely metrizable. By Lemma 2.2 $X = L \oplus R$, where L is locally compact without isolated points and R has the property that every compact subset of R is nowhere dense in R . By Theorem 1.5, R is either empty or homeomorphic to ω^ω and L (if non-empty) is homeomorphic either to 2^ω or $2^\omega \setminus \{0\}$ depending on whether it is compact or not. \square

A natural question is whether the above results can be extended beyond analytic or Borel sets. The answer depends on set theoretic assumptions. For possible extensions note that all arguments presented so far use only the validity of Theorems 1.1, 1.3, 1.4 and only countable Axiom of Choice, a consequence of the Axiom of Dependent Choice.

Corollary 2.7. (i) (PD) *Every projective CDH space is completely metrizable.*
(ii) (AD) *All separable metric CDH spaces are completely metrizable.*

So in particular, it is consistent with ZF that every zero-dimensional metric CDH space without isolated point is homeomorphic to one of the following spaces: 2^ω , ω^ω , $2^\omega \setminus \{0\}$, $\omega^\omega \oplus 2^\omega$ and $\omega^\omega \oplus 2^\omega \setminus \{0\}$.

To conclude the section we show that the Theorem 2.3 and Corollary 2.4 are consistently sharp by proving the following:

Theorem 2.6. (MA + $\neg\text{CH}$ + $\omega_1 = \omega_1^L$) *Let X be an \aleph_1 -dense subset of 2^ω . Then: (i) X is a co-analytic meager in itself CDH space. (ii) $2^\omega \setminus X$ is an analytic completely Baire CDH space which is not completely metrizable.*

Proof. A theorem of Martin and Solovay (see [Mi]) states that, assuming MA + $\neg\text{CH}$ + $\omega_1 = \omega_1^L$, every set of reals of size \aleph_1 is co-analytic. MA implies that X is meager in itself. It is easy to see that $2^\omega \setminus X$ is completely Baire and not completely metrizable (and of course analytic).

The fact that both X and $2^\omega \setminus X$ are CDH follows directly from Lemma 3.1 of [BB]. \square

3. PRODUCTS OF CDH SPACES

Theorem 2.6 can be used to see that products of CDH spaces need not be CDH. In fact, if X is a meager in itself CDH metric space, it is easy to see that $X \times \mathbb{R}$ is not CDH¹. On the other hand, infinite products of spaces which are not CDH can be

¹The authors are not aware of a ZFC example of two metric CDH spaces whose product is not CDH.

CDH, an example being the Hilbert cube $[0, 1]^\omega$ ([Fo]). Lawrence [La] showed that X^ω is homogeneous, for every $X \subseteq 2^\omega$ (see also [DP]) answering half of Question 388. of [ZH3]. The other half asks for which $X \subseteq 2^\omega$ is X^ω CDH. It was known that not for all as Fitzpatrick and Zhou in [FZ2] showed that \mathbb{Q}^ω is not CDH, where \mathbb{Q} denotes the space of rational numbers. In this section we characterize those Borel subsets of 2^ω whose power is CDH.

Theorem 3.1. *Let X be a separable metric space such that X^ω is CDH. Then X is a Baire space.*

Proof. The proof of this theorem is quite analogous to the proof of Claim 2 of Theorem 2.3. Suppose that X has at least two elements. It suffices to note, that (1) if X is not Baire then X^ω is meager in itself, and (2) Every open subset of X^ω contains a copy of 2^ω . \square

Theorem 3.2. *Let $X \subseteq 2^\omega$ be Borel. Then the following are equivalent:*

- (1) X^ω is CDH
- (2) X is G_δ in 2^ω ,
- (3) $|X^\omega| = 1$ or X^ω is homeomorphic to 2^ω or X^ω is homeomorphic to ω^ω .

Proof. (1) implies (2) by Theorem 2.3 as X^ω is Borel if and only if X is and, moreover, X^ω is completely metrizable if and only if X is.

To see that (2) implies (3) note that if X is G_δ in 2^ω then X^ω is completely metrizable. Moreover, if X is zero-dimensional then so is X^ω and X^ω does not contain any isolated points. Suppose that $|X^\omega| > 1$. Now, if X is compact then so is X^ω , hence, X^ω is homeomorphic to 2^ω by Theorem 1.5 (i). If X is not compact then all compact subsets of X^ω are nowhere dense and X^ω is homeomorphic to ω^ω by Theorem 1.5 (ii).

(3) implies (1), as both 2^ω and ω^ω are CDH. \square

Just like in the previous section this theorem can be strengthened assuming PD or AD. The following question, however, remains open.

Question 3.2. *Is there a non- G_δ subset of 2^ω such that X^ω is CDH?*

We will conclude this section and the paper by considering uncountable products. Recall that a family $\mathcal{F} \subseteq [\omega]^\omega$ is *centered* if every non-empty finite subfamily of \mathcal{F} has an infinite intersection. An infinite set $A \subseteq \omega$ is a *pseudo-intersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ if $A \setminus F$ is finite for every $F \in \mathcal{F}$. The cardinal invariant \mathfrak{p} is defined as the minimal cardinality of a centered family $\mathcal{F} \subseteq [\omega]^\omega$ which has no infinite pseudo-intersection.

Steprāns and Zhou in [SZ] showed that 2^κ is CDH for every $\kappa < \mathfrak{p}$ and asked whether $2^\mathfrak{p}$ is provably not CDH. We show that it follows from known results that the answer is positive.

Theorem 3.3. $\mathfrak{p} = \min\{\kappa : 2^\kappa \text{ is not CDH}\}.$

Proof. The fact that $\min\{\kappa : 2^\kappa \text{ is not CDH}\} \leq \mathfrak{p}$ was proved in [SZ]. In [Ma] and [HS] it is shown that there is a countable dense set $D \subseteq 2^\mathfrak{p}$ and a point $x \in 2^\mathfrak{p}$ such that no sequence in D converges to x . On the other hand, it is easy

to construct a countable dense set $C \subseteq 2^{\mathfrak{p}}$ such that for every $c \in C$ there is a sequence $\langle c_n : n \in \omega \rangle \subseteq C \setminus \{c\}$ converging to c .

Now, notice that there is no homeomorphism of $2^{\mathfrak{p}}$ sending C to $D \cap \{x\}$ as if $c = h^{-1}(x)$ and $\langle c_n : n \in \omega \rangle \subseteq C \setminus \{c\}$ a sequence converging to c , then the sequence $\langle h(c_n) : n \in \omega \rangle$ does not converge to x contradicting continuity of h . \square

Acknowledgments. The work contained in this paper is part of the second author's Master's thesis at the Universidad Michoacana de San Nicolás de Hidalgo, written under the supervision of the first author. The first author wishes to thank A. Louveau, I. Farah, J. van Mill and S. Todorćević for bibliographical information and fruitful discussion.

REFERENCES

- [BB] S. Baldwin and R. E. Beaudoin, *Countable dense homogeneous spaces under Martin's axiom*, Israel J. Math. **65** (1989), 153–164.
- [Be] R. B. Bennett, *Countable dense homogeneous spaces*, Fun. Math **74** (1972), 189–194.
- [Br] L. E. J. Brower, *Some Remarks on the coherence type η* , Proc. Akad. Amsterdam **15** (1256–1263), 1912.
- [DP] A. Dow and E. Pearl, *Homogeneity in Powers of zero-dimensional, first-countable spaces*, Proc. AMS **125** (1997), 2503–2510.
- [Fi] B. Fitzpatrick Jr., *A note on countable dense homogeneity*, Fund. Math. **75** (1972), 3–4.
- [FL] B. Fitzpatrick, Jr. and N. F. Lauer, *Densely homogeneous spaces. I*, Houston J. Math. **13** (1987), 19–25.
- [FZ1] B. Fitzpatrick Jr. and H.-X. Zhou, *Densely homogeneous spaces II*, Houston J. Math. **14** (1988), 57–68.
- [FZ2] B. Fitzpatrick Jr. and H.-X. Zhou, *Countable dense homogeneity and the Baire property*, Topology and its Applications **43** (1992), 1–14.
- [FZ3] B. Fitzpatrick Jr. and H.-X. Zhou, *Some Open Problems in Densely Homogeneous Spaces*, in Open problems in Topology (ed. J. van Mill and M. Reed), 1984, pp. 251–259, North-Holland, Amsterdam.
- [Fo] M. Fort, *Homogeneity of infinite products of manifolds with boundary*, Pacific J. Math **12** (1962), 879–884.
- [Fr] M. Fréchet, *Les dimension d'un ensemble abstrait*, Math. Ann **68** (1910), 145–168.
- [HS] M. Hrušák and J. Steprans, *Cardinal invariants related to sequential separability*, Surikaiseikenkiusho Kokyuroku **1202** (2001), 66–74.
- [Ka] A. Kanamori, *The Higher Infinite*, 1994, Springer-Verlag.
- [Ke] A. S. Kechris, *Classical Descriptive Set Theory*, 1995, Springer-Verlag.
- [KLW] A. S. Kechris, A. Louveau and W. H. Woodin, *The Structure of σ -ideals of Compact Sets*, Trans. AMS **301** (1987), 263–288.
- [Ku] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, 1990, North Holland.
- [La] B. Lawrence, *Homogeneity in powers of subspaces of the real line*, Trans. AMS **350** (1998), 3055–3064.
- [Ma] M. V. Matveev, *Cardinal \mathfrak{p} and a theorem of Pelczyński*, (preprint).
- [vM1] J. van Mill, *Strong local homogeneity does not imply countable dense homogeneity*, Proc. AMS **84** (1982), 143–148.
- [vM2] J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, 2001, North Holland.
- [Mi] A. W. Miller, *Descriptive Set Theory and Forcing*, 1995, Springer, Lecture Notes in Logic 4.

- [Sa] W. L. Saltzman, *Concerning the existence of a connected, countable dense homogeneous subset of the plane which is not strongly locally homogeneous*, Topology Proceedings **16** (1991), 137–176.
- [SZ] J. Steprans, H.-X. Zhou, *Some Results on CDH Spaces*, Topology and its Applications **28** (1988), 147–154.
- [Zh] H.-X. Zhou, *Two applications of set theory to homogeneity*, Questions Answers Gen. Topology **6** (1988), 49–56.

Instituto de matemáticas, UNAM Unidad Morelia

A. P. 61-3

Xangari

C. P. 58089, Morelia, Mich., México

michael@matmor.unam.mx , *bzamora@matmor.unam.mx*